From One Ternary Table to a Complete Indexed Universe of Binary Operations

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A finite symbolic system is developed from one binary operation w on the carrier A = {0,1,2}, specified by the table 201/122/010. Beginning only with variables, substitution, composition, and complete extensional evaluation, the system generates all 3^9 = 19,683 binary operations $A^2 \to A$. Seed-derived ternary contexts C(z,x,y) act cellwise on operation tables and yield reversible transformations of the complete operation universe. Five generated founder contexts connect all 19,683 tables, producing 314,928 reversible programs with exactly sixteen program representatives per visible operation. A deterministic breadth-first construction selects one shortest canonical program for every table, with maximum founder length nine. The remaining sixteenfold ambiguity is represented by four silent binary coordinates. A compact multiscale navigation index records operation birth ranks, frontier masks, and cumulative checkpoints, permitting exact random-access reconstruction of canonical programs and rewrite corrections without storing the complete program transversal. The resulting 48-KiB navigator demonstrates how one anonymous ternary table can generate, connect, quotient, index, and navigate an entire finite operation universe.

Keywords: ternary carrier, binary operation, finite algebra, term generation, reversible context, canonical program, breadth-first search, operation universe, symbolic discovery, exact verification, quotient memory, multiscale navigation.

Code: https://github.com/DougYouvan/youvan-ai-v0.34.0-minimal-seed-multiscale-birth-rank-navigation-index-

 https://orcid.org/0009-0004-4206-4253

Google Drive:
https://drive.google.com/drive/folders/15Kd8bUjm7zVaKdVIwbPhqjjV68aUZQvc?usp=sharing

Outline

1. Primitive Seed

Let

$A = \{0,1,2\}$.

Define one binary operation

$w : A \times A \to A$

by the table

w(a,b)	**b = 0**	**b = 1**	**b = 2**
a = 0	2	0	1
a = 1	1	2	2
a = 2	0	1	0

Equivalently,

w = 201/122/010.

Thus,

$w(0,0) = 2$,
$w(0,1) = 0$,
$w(0,2) = 1$,

$w(1,0) = 1$,
$w(1,1) = 2$,
$w(1,2) = 2$,

$w(2,0) = 0$,
$w(2,1) = 1$,
$w(2,2) = 0$.

No algebraic interpretation is assigned to w in advance. It is not initially declared to be associative, commutative, invertible, ordered, logical, geometric, or topological. Its entries are treated only as the complete specification of a finite operation.

The primitive invention language contains:

1. variables $x_0, x_1, \ldots$;
2. the operation symbol w;

3. recursive term formation;
4. substitution;
5. complete evaluation on A;
6. extensional equality.

If s and t are terms in n variables, define

$s \equiv t$

when

$s(a_1,\dots,a_n) = t(a_1,\dots,a_n)$

for every point of A^n.

Because A is finite, extensional equality is decidable by exhaustive evaluation. For binary terms, only nine evaluations are required.

The seed therefore supplies both syntax and exact semantics:

w + variables → terms → finite functions.

No corpus of known mathematics is used. No library of groups, rings, lattices, semigroups, or logical connectives is available to the invention layer.

The developmental constraint is:

Every invented operation or representation must be expressible as a term constructed from w, variables, substitution, and previously admitted seed-derived terms.

The initial mathematical world is consequently minimal:

$\mathcal{S}_0 = (A,w)$.

All later structures arise from repeated internal use of this single finite operation.

2. Term-Generated Binary Operations

Let T_2 be the set of terms generated recursively from variables x and y by

$x,y \in T_2$,

and

$s,t \in T_2 \Rightarrow w(s,t) \in T_2$.

Each term $t \in T_2$ induces a binary operation

$t^A : A^2 \to A$

by complete evaluation in the seed algebra $\mathcal{S}_0$.

Examples include

x,

y,

w(x,y),

w(w(x,y),x),

w(w(x,x),w(y,x)).

Distinct syntactic terms may induce the same operation. Define the extensional quotient

$\mathcal{T}_2 = T_2 / \equiv$,

where

$s \equiv t \Leftrightarrow \forall(a,b) \in A^2, s^A(a,b)=t^A(a,b)$.

Each equivalence class is represented computationally by its nine-value table

[t(0,0),t(0,1),…,t(2,2)].

Because each of the nine outputs lies in A, the total number of possible binary operations is

$|A|$^{$|A|^2$} = 3^9 = 19,683.

Define term height recursively by

h(x)=h(y)=0,

h(w(s,t))=1+max{h(s),h(t)}.

The number of distinct binary term functions obtained by bounded height is

N_0 = 2,

N_1 = 6,

N_2 = 38,

N_3 = 1,256,

N_4 = 19,681.

Thus the growth is not merely syntactic. Extensional quotienting converts a rapidly expanding term language into a finite sequence of genuinely distinct operations.

At height four, only two of the 19,683 possible binary operations remain absent. Their table codes are

6338,

11645.

Equivalently, their tables are

202/002/220,

220/222/021.

These two operations are not absent from the full seed-generated world. They require one additional representation step, described below.

The direct term-generation process therefore establishes

$|\mathcal{T}_2^{\{\leq 4\}}| = 19{,}681$,

where $\mathcal{T}_2^{\{\leq 4\}}$ denotes the extensional classes represented by terms of height at most four.

This near-completion is already structurally significant:

$3^9 - |\mathcal{T}_2^{\{\leq 4\}}| = 2$.

Hence one anonymous ternary operation generates all but two binary operations after only four levels of recursive composition.

The term universe is finite extensionally but infinite syntactically. Once all 19,683 functions have appeared, further terms do not create new binary operations; they create additional representations of existing operations.

Thus the developmental transition is

new syntax → new functions → saturation → representational redundancy.

The later stages of the construction exploit this redundancy rather than discard it.

3. Completion of the 19,683-Operation Universe

The two missing operations are obtained by applying seed-derived transformations to operations already present at height four.

Define

$g_0(z,x,y) = w(z,x)$,

$g_1(z,x,y) = w(z,y)$.

For any binary operation $f : A^2 \to A$, these expressions induce new binary operations

$g_0f = w(f(x,y),x)$,

$g_1f = w(f(x,y),y)$.

Both transformations use only the original seed operation and variables. No external operation is introduced.

Let

m_0 = 202/002/220,

m_1 = 220/222/021

be the two tables absent from the direct height-four term universe.

The exhaustive construction yields height-four terms p and q such that

$g_1[p] = m_0$,

$g_0[q] = m_1$.

Substitution gives ordinary seed terms

$m_0(x,y) = w(p(x,y),y)$,

$m_1(x,y) = w(q(x,y),x)$.

Both resulting expressions have height five and contain seventeen operation nodes.

Theorem 1. Functional completeness of the seed

Every binary operation

$f : A^2 \to A$

is represented by a term over the single operation w.

Proof

There are exactly $3^9 = 19{,}683$ binary operations on A.

Direct term enumeration through height four produces 19,681 distinct operations. The two absent operations are m_0 and m_1. Each is obtained by one additional composition with w from an already generated height-four operation.

Therefore all 19,683 binary operations are seed-generated by height at most five. ■

Hence

$Clo_2(w) = A^{A^2}$,

where $Clo_2(w)$ denotes the binary part of the term clone generated by w.

Equivalently, w is functionally complete for binary operations on A.

The result may be summarized by

one 3 × 3 table

- two variables
- recursive composition
  = all 3^9 binary tables.

The construction is stronger than enumeration. Enumeration merely lists all possible tables. The present result supplies a term witness for every table.

Thus each operation f has a finite derivation

w → terms → f.

The height bound is

$\forall f : A^2 \to A, \exists t$ such that $t^A = f$ and $h(t) \le 5$.

The distribution is sharply concentrated:

19,681 operations appear by height four,

2 operations first appear at height five.

Therefore the seed reaches 99.9898% of the binary universe directly by height four and completes the universe with one additional layer.

After completion, continued composition cannot enlarge the binary function set. It can only create new programs representing already generated operations.

This marks the transition from functional generation to representational organization:

complete operation universe
→ multiple seed terms per operation
→ transformations among representations
→ canonical program structure.

4. Seed-Derived Representation Contexts

Binary terms generate operations. Ternary terms generate transformations of operations.

Let

$$C(z, x, y)$$

be a term over the same seed operation w, where $z$is a placeholder for the current table entry and $x, y$are the input coordinates.

For any binary operation

$$f: A^2 \to A,$$

define the transformed operation

$$C[f](x, y) = C(f(x, y), x, y).$$

Thus a single ternary term induces a map

$$\hat{C}: A^{A^2} \to A^{A^2}$$

on the complete 19,683-operation universe.

For example,

$$C(z, x, y) = w(z, x)$$

induces

$$\hat{C}(f)(x, y) = w(f(x, y), x),$$

while

$$D(z, x, y) = w(z, y)$$

induces

$$\widehat{D}(f)(x, y) = w(f(x, y), y).$$

These transformations are not supplied externally. They are terms in the same primitive language that generated the operations themselves.

Define context height by

$$h(z) = h(x) = h(y) = 0,$$
$$h(w(s, t)) = 1 + \max\{h(s), h(t)\}.$$

Two contexts are extensionally equal when

$$C(a, b, c) = D(a, b, c)$$

for all

$$(a, b, c) \in A^3.$$

Since

$$\mid A^3 \mid = 27,$$

context equality is decided by comparing 27 outputs.

The number of distinct ternary context functions generated through bounded height is

$$M_0 = 3,$$
$$M_1 = 12,$$
$$M_2 = 147,$$
$$M_3 = 21{,}014.$$

Among the height-three contexts, 19,758 depend nontrivially on the placeholder z. These are genuine candidate representations of an existing operation rather than coordinate-only replacements.

A context is pointwise reversible when, for every fixed pair (x, y), the unary map

$$z \mapsto C(z, x, y)$$

is a permutation of A.

Pointwise reversibility implies that the induced map on operation tables is bijective, because every table cell is transformed independently by a reversible unary map.

Proposition 2. Cellwise reversibility

If $C$is pointwise reversible, then

$$\hat{C}: A^{A^2} \to A^{A^2}$$

is a permutation of the 19,683 binary operations.

Proof

For each $(x, y) \in A^2$, let

$$\pi_{x,y}(z) = C(z, x, y).$$

By hypothesis, each $\pi_{x,y}$is a permutation of A. Therefore each transformed cell uniquely determines its original value through $\pi_{x,y}^{-1}$.

Applying the nine inverse permutations cellwise reconstructs the original table uniquely. Hence $\hat{C}$is bijective. ■

The exhaustive context search found

$$799$$

pointwise reversible contexts through height three.

Among these,

$$68$$

nonidentity contexts possess inverses within the bounded generated context universe, and

$$56$$

are self-inverse.

One selected self-inverse context is

$$C(z, x, y) = w(w(z, x), w(x, x)).$$

It satisfies

$$C(C(z, x, y), x, y) = z$$

for all 27 assignments of (z, x, y).

Consequently,

$$\hat{C}^2 = \mathrm{id}$$

on all 19,683 binary operations.

Its induced permutation has cycle decomposition

$$1^1\, 2^{9841}.$$

That is, it has one fixed operation and 9,841 transposed pairs.

The unique fixed table is

$$000/222/222.$$

This establishes the first representation-theoretic transition:

seed term ⟶ reversible context ⟶ permutation of the complete operation universe.

The original seed therefore manufactures not only objects but also exact reversible ways of viewing and transforming those objects.

5. Reversible Founder Transformations

Let

$$\mathcal{O} = A^{A^2}$$

denote the complete set of 19,683 binary operations on A.

A reversible context $g$induces a permutation

$$\hat{g} \in \mathrm{Sym}(\mathcal{O}).$$

For reversible contexts $g$and h, define composition by

$$(g \circ h)[f] = g[h[f]].$$

The identity context is

$$e(z, x, y) = z.$$

If $g$is reversible, then there exists g^{-1}such that

$$g^{-1} \circ g = e = g \circ g^{-1}.$$

The first two recurring contexts selected by minimum term complexity are

$$g_0(z, x, y) = w(z, x),$$
$$g_1(z, x, y) = w(z, y).$$

Both satisfy

$$g_0^6 = e, g_1^6 = e.$$

Hence

$$g_0^{-1} = g_0^5, g_1^{-1} = g_1^5.$$

Their closure contains

$$324$$

distinct reversible contexts:

$$\mathcal{E}_2 = \langle g_0, g_1 \rangle, \mid \mathcal{E}_2 \mid = 324.$$

Acting on the original seed w, these contexts generate 324 distinct operation tables:

$$|\mathcal{E}_2 \cdot w| = 324.$$

Therefore the map

$$\Phi_w : \mathcal{E}_2 \to \mathcal{E}_2 \cdot w, \Phi_w(g) = g[w],$$

is bijective.

Proposition 3. Seed self-indexing

Every operation in the orbit $\mathcal{E}_2 \cdot w$uniquely identifies the context that generated it from w.

Proof

The domain and image each contain 324 elements, and direct evaluation shows that distinct contexts produce distinct transformed seed tables. Hence $\Phi_w$is bijective. ■

Thus the seed orbit is a table-valued index of its own transformation programs.

For any two orbit states $a, b \in \mathcal{E}_2 \cdot w$, there exists a unique context $t_{a,b} \in \mathcal{E}_2$satisfying

$$t_{a,b}[a] = b.$$

Indeed, if

$$a = g[w], b = h[w],$$

then

$$t_{a,b} = h \circ g^{-1}.$$

The complete transition system contains

$$324^2 = 104{,}976$$

exact ordered transitions.

The initial two-founder ecology does not act transitively on all of $\mathcal{O}$. Its action partitions the complete operation universe into 69 orbits:

$$\mathcal{O}/\mathcal{E}_2 = \{Q_1, \ldots, Q_{69}\}.$$

Their sizes are

81once,
162fifteen times,
324fifty-three times.

Accordingly,

$$81 + 15(162) + 53(324) = 19{,}683.$$

Each orbit supplies a canonical descendant operation. Applying the same seed-term context search to these 69 descendants produces additional reversible founders.

A distinguished descendant generates three contexts whose closure contains

$$17{,}496$$

programs acting on an orbit of size

$$2{,}187.$$

Two further founders, arising from two other seed-generated descendants, connect the remaining orbit classes.

Let

$$G = \{g_0, g_1, g_2, g_3, g_4\}$$

be the resulting five-context founder set, and let

$$\mathcal{E} = \langle G \rangle.$$

Exact closure gives

$$|\ \mathcal{E}\ | = 314{,}928.$$

Every element of $\mathcal{E}$is a context term over the original operation w. No transformation external to the seed language is admitted.

Theorem 4. Founder-generated transitivity

The five founder contexts act transitively on the complete operation universe:

$$\mathcal{E} \cdot w = \mathcal{O}.$$

Proof

Exact closure under the five founders generates 314,928 reversible contexts. Applying these contexts to the original seed reaches all 19,683 binary operation tables. Therefore the orbit of $w$under $\mathcal{E}$is the complete set $\mathcal{O}$. ■

Hence, for every pair

$$a, b \in \mathcal{O},$$

there exists a seed-derived reversible program $p \in \mathcal{E}$such that

$$p[a] = b.$$

The finite operation universe is therefore not merely generated. It is reversibly connected by a five-symbol transformation language manufactured from the same primitive table.

6. Complete Connectivity

Let

$$\mathcal{O} = A^{A^2}$$

be the set of all 19,683 binary operations on $A = \{0,1,2\}$, and let

$$\mathcal{E} = \langle g_0, g_1, g_2, g_3, g_4 \rangle$$

be the reversible context system generated by the five seed-derived founders.

Define a directed labeled graph

$$\Gamma = (V, E)$$

by

$$V = \mathcal{O},$$

and

$$(f, g_i[f]) \in E$$

for every $f \in \mathcal{O}$and every founder g_i.

Because each founder is reversible, every directed edge has a reverse path. Thus Γmay be regarded as an undirected connected transition graph with five labeled move types.

Theorem 5. Complete reversible connectivity

For every pair of operations

$$f, h \in \mathcal{O},$$

there exists a finite founder word

$$p = g_{i_k} \circ \cdots \circ g_{i_1}$$

such that

$$p[f] = h.$$

Proof

The orbit of the original seed under $\mathcal{E}$is all of $\mathcal{O}$. Hence there exist programs $u, v \in \mathcal{E}$satisfying

$$u[w] = f, v[w] = h.$$

Since $u$is reversible,

$$u^{-1}[f] = w.$$

Therefore

$$(v \circ u^{-1})[f] = h.$$

The composite $v \circ u^{-1}$is again a seed-derived reversible context. ■

The transition graph is therefore connected:

$$\text{components}(\Gamma) = 1.$$

The result may be written schematically as

$$\forall f, h \in \mathcal{O}, f \rightsquigarrow h.$$

Here $\rightsquigarrow$ denotes a finite sequence of founder applications.

The connected ecology contains

$$|\mathcal{E}| = 314{,}928$$

distinct reversible programs. Since the visible operation universe contains 19,683 states,

$$\frac{|\mathcal{E}|}{|\mathcal{O}|} = \frac{314{,}928}{19{,}683} = 16.$$

Thus exactly sixteen programs carry the original seed to each visible operation.

For $f \in \mathcal{O}$, define its program fiber by

$$F_f = \{p \in \mathcal{E}: p[w] = f\}.$$

Then

$$|F_f| = 16$$

for every $f \in \mathcal{O}$.

Hence the evaluation map

$$\pi: \mathcal{E} \to \mathcal{O}, \pi(p) = p[w],$$

is a uniform sixteen-to-one surjection.

The program world therefore decomposes as

$$314{,}928 = 19{,}683 \times 16.$$

This separates two kinds of information:

visible state = operation table,
hidden state = choice among sixteen equivalent programs.

Let

$$K = \{p \in \mathcal{E}: p[w] = w\}$$

be the set of programs that fix the original seed. Then

$$| K |= 16.$$

Every program fiber is a translated copy of K. If $s_f$is any program satisfying

$$s_f[w] = f,$$

then

$$F_f = s_f K.$$

Thus each reversible program admits a factorization

$$p = s_f k, k \in K.$$

The visible operation is determined by s_f; the residual program $k$acts silently on the seed.

Proposition 6. Uniform program quotient

The complete program ecology projects onto the complete operation universe with constant fiber size sixteen:

$$\mathcal{E}/K \cong \mathcal{O}$$

as a finite transitive action space.

The quotient identifies programs that produce the same visible operation from the seed.

Complete connectivity therefore yields two simultaneous structures:

$$\mathcal{E}$$

as the reversible program universe, and

$$\mathcal{O}$$

as its visible quotient.

The next problem is to choose one preferred representative from each fiber F_f. This produces a canonical shortest program for every operation and converts the connected universe into an indexed navigable system.

7. Canonical Shortest Programs

Let

$$G = \{g_0, g_1, g_2, g_3, g_4\}$$

be the five seed-derived founder contexts.

A founder word is a finite sequence

$$\alpha = i_1 i_2 \cdots i_k, i_j \in \{0,1,2,3,4\}.$$

It induces the program

$$g_\alpha = g_{i_k} \circ \cdots \circ g_{i_1}.$$

The word length is

$$|\alpha| = k.$$

For each operation $f \in \mathcal{O}$, define its founder distance from the seed by

$$d(f) = \min\{|\alpha| : g_\alpha[w] = f\}.$$

Because the founder action is transitive, $d(f)$is finite for every operation.

A deterministic breadth-first search begins with

$$w$$

at depth zero and repeatedly applies the founders in the fixed order

$$g_0, g_1, g_2, g_3, g_4.$$

An operation is recorded only when it first appears.

The first-discovery word of f is denoted

$$s(f).$$

By breadth-first construction,

$$|s(f)| = d(f).$$

Thus $s(f)$ is a shortest founder program carrying the original seed to f.

When several shortest words exist, the fixed founder order selects one deterministically.

Definition 7. Canonical section

The map

$$s: \mathcal{O} \to \mathcal{E}$$

is defined by assigning to each operation its first breadth-first program:

$$s(f)[w] = f.$$

It is a section of the evaluation map π, because

$$\pi(s(f)) = f.$$

The section chooses one representative from each sixteen-element program fiber.

Exact enumeration gives

$$|\mathrm{im}(s)| = 19{,}683.$$

The maximum canonical founder length is

$$\max_{f \in \mathcal{O}} d(f) = 9.$$

Therefore every operation is reached by a preferred program using at most nine founder applications:

$$\forall f \in \mathcal{O}, | s(f) | \leq 9.$$

The total number of founder symbols in the complete canonical transversal is

$$\sum_{f \in \mathcal{O}} | s(f) | = 137{,}885.$$

The mean canonical length is therefore

$$\frac{137{,}885}{19{,}683} \approx 7.005.$$

The canonical section converts the complete operation universe into a rooted finite graph:

$$w \longrightarrow \text{depth-one states} \longrightarrow \cdots \longrightarrow \text{depth-nine states.}$$

Every nonseed operation f has:

1. a canonical parent $p(f)$;
2. a canonical founder label $i(f)$;

such that

$$f = g_{i(f)}[p(f)]$$

and

$$d(p(f)) = d(f) - 1.$$

Repeated parent recovery gives

$$f \to p(f) \to p^2(f) \to \cdots \to w.$$

Reversing the corresponding founder labels reconstructs $s(f)$.

Proposition 8. Exact shortest-program recovery

The canonical parent relation determines a unique shortest selected program for every operation.

Proof

Each nonseed operation is first discovered from one previously discovered state by one founder. Its parent therefore lies one breadth-first layer earlier. Repeated parent application strictly decreases depth and must terminate at the seed. The recovered founder sequence has length $d(f)$, so it is shortest. Deterministic first discovery makes the selected sequence unique. ■

The section is canonical relative to three frozen choices:

$$\text{seed}, \text{founder set}, \text{founder order}.$$

It is not claimed to be invariant under arbitrary changes of presentation.

Nevertheless, within the declared construction, it supplies an exact normal representative for every visible operation:

$$f \leftrightarrow s(f).$$

Any reversible program $p \in \mathcal{E}$may therefore be compared with the canonical representative of its visible output:

$$f = p[w].$$

The difference between $p$and $s(f)$is a residual program fixing the seed:

$$r(p) = s(f)^{-1} \circ p.$$

Since

$$r(p)[w] = w,$$

we have

$$r(p) \in K.$$

Hence every program has the unique normal form

$$p = s(\pi(p))\, r(p), r(p) \in K.$$

The canonical section therefore separates the complete reversible program world into:

shortest visible program + silent residual state.

8. Sixteenfold Program Redundancy

Let

$$\pi: \mathcal{E} \to \mathcal{O}$$

be the evaluation map

$$\pi(p) = p[w].$$

For every visible operation $f \in \mathcal{O}$, the fiber

$$F_f = \pi^{-1}(f)$$

contains exactly sixteen reversible programs.

Hence

$$|\mathcal{E}| = \sum_{f \in \mathcal{O}} |F_f| = 19{,}683 \times 16 = 314{,}928.$$

The redundancy is uniform:

$$|F_f| = 16 \text{for all } f \in \mathcal{O}.$$

Define the silent residual set

$$K = \pi^{-1}(w).$$

Thus

$$K = \{k \in \mathcal{E} : k[w] = w\}.$$

Exact enumeration gives

$$|K| = 16.$$

For any $f \in \mathcal{O}$, let $s(f)$denote its canonical shortest program. Then every program in $F_f$has the form

$$s(f) \circ k, k \in K.$$

Conversely, every such composite lies in F_f, because

$$(s(f) \circ k)[w] = s(f)[k[w]] = s(f)[w] = f.$$

Therefore

$$F_f = s(f)K.$$

Theorem 9. Canonical residual factorization

Every program $p \in \mathcal{E}$admits a unique factorization

$$p = s(f) \circ k,$$

where

$$f = \pi(p)$$

and

$$k \in K.$$

Proof

Set

$$f = \pi(p)$$

and define

$$k = s(f)^{-1} \circ p.$$

Then

$$k[w] = s(f)^{-1}[p[w]] = s(f)^{-1}[f] = w,$$

so $k \in K$. Hence

$$p = s(f) \circ k.$$

For uniqueness, suppose

$$s(f) \circ k_1 = s(f) \circ k_2.$$

Left composition by $s(f)^{-1}$gives

$$k_1 = k_2.$$

Thus the factorization is unique. ■

The complete program universe therefore admits the exact set-theoretic decomposition

$$\mathcal{E} \cong \mathcal{O} \times K.$$

This is not merely a counting identity. The decomposition supplies a normal form for every reversible program:

$$p \leftrightarrow (\pi(p), r(p)),$$

where

$$r(p) = s(\pi(p))^{-1} \circ p.$$

The first coordinate is visible:

$$\pi(p) \in \mathcal{O}.$$

The second coordinate is silent on the original seed:

$$r(p) \in K.$$

The residual set has the exact size

$$16 = 2^4.$$

Exhaustive composition shows:

$$k^2 = e$$

for every nonidentity $k \in K$, and

$$k \circ \ell = \ell \circ k$$

for every $k, \ell \in K$.

Thus every nonidentity residual is self-inverse, and all residuals commute.

Four residual generators are necessary and sufficient. Let

$$b_1, b_2, b_3, b_4 \in K$$

be one minimum generating basis. Then every residual has a unique representation

$$k = b_1^{\varepsilon_1} b_2^{\varepsilon_2} b_3^{\varepsilon_3} b_4^{\varepsilon_4},$$

where

$$\varepsilon_i \in \{0,1\}.$$

Therefore

$$K \cong (\mathbb{Z}/2\mathbb{Z})^4.$$

This algebraic name is assigned by the verifier after discovery. The invention layer discovers only:

16 residual states,

15 nonidentity involutions,
pairwise commutation,
minimum generator number 4.

The residual coordinate of a program may therefore be written as the four-bit vector

$$\varepsilon(p) = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \in \{0,1\}^4.$$

Residual composition becomes bitwise addition modulo two:

$$\varepsilon(k \circ \ell) = \varepsilon(k) \oplus \varepsilon(\ell).$$

The complete reversible program world consequently has the normal form

$$p \leftrightarrow (f, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4),$$

with

$$f \in \mathcal{O}.$$

The visible operation requires one of 19,683 labels. The residual ambiguity requires exactly four additional bits.

Hence the sixteenfold redundancy is not discarded. It becomes a compact hidden memory attached uniformly to every visible operation.

9. Four-Bit Silent Residual Memory

The residual system

$$K = \{k \in \mathcal{E}: k[w] = w\}$$

contains sixteen programs. Each acts trivially on the visible seed state but may act nontrivially on other operations.

Thus residual memory is invisible at the root:

$$k[w] = w,$$

yet dynamically significant elsewhere:

$$k[f] \neq f$$

for suitable $f \in \mathcal{O}$.

Choose a minimum residual basis

$$B = \{b_1, b_2, b_3, b_4\}.$$

Every $k \in K$ has a unique coordinate vector

$$\chi(k) = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \in \{0,1\}^4$$

such that

$$k = b_1^{\varepsilon_1} b_2^{\varepsilon_2} b_3^{\varepsilon_3} b_4^{\varepsilon_4}.$$

The identity residual has coordinate

$$\chi(e) = (0,0,0,0).$$

The fifteen nonidentity residuals realize the remaining fifteen binary vectors.

Residual composition satisfies

$$\chi(k \circ \ell) = \chi(k) \oplus \chi(\ell),$$

where $\oplus$ is coordinatewise addition modulo two.

Hence

$$\chi: K \to \{0,1\}^4$$

is a bijection preserving composition.

The four residual bits are recoverable from local context behavior.

A context acts at each table position (x, y) through the unary map

$$z \mapsto C(z, x, y).$$

For reversible contexts, each local action is one of the six permutations of A.

The system searched all subsets of the nine positions

$$A^2 = \{(0,0), (0,1), \dots, (2,2)\}$$

for the smallest subset distinguishing all sixteen residuals.

The minimum size is four.

One selected minimum basis of observation sites is

$$S = \{(0,0), (0,2), (1,0), (2,0)\}.$$

At each selected site, the residual action assumes only two possible values:

$$\mathrm{id}_A$$

or one fixed transposition of two elements of A.

Associate bit 0with the identity local action and bit 1with the transposition. Then each residual produces a four-bit observation vector

$$\rho_S(k) \in \{0,1\}^4.$$

Exact enumeration yields

$$\rho_S(K) = \{0,1\}^4.$$

Therefore

$$\rho_S: K \to \{0,1\}^4$$

is bijective.

Proposition 10. Minimum local residual reconstruction

Four local context observations are sufficient and necessary to identify every residual program.

Proof

The selected four-site set produces sixteen distinct signatures, so four observations are sufficient.

At each selected-type site, only two residual actions occur. Therefore three such observations distinguish at most

$$2^3 = 8$$

states, fewer than the sixteen residuals. Hence three observations cannot suffice. ■

There are

$$54$$

distinct minimum four-site observation bases.

Thus the residual memory admits multiple equivalent local coordinate systems.

For a complete program p, define

$$f = \pi(p)$$

and

$$k = s(f)^{-1} \circ p.$$

Its exact normal coordinate is

$$N(p) = (f, \rho_S(k)).$$

This gives a bijection

$$N: \mathcal{E} \to \mathcal{O} \times \{0,1\}^4.$$

The information requirement is exact. Since

$$2^{18} < 314{,}928 < 2^{19},$$

at least nineteen bits are needed to identify an arbitrary program.

The visible operation requires fifteen bits because

$$2^{14} < 19{,}683 < 2^{15}.$$

The residual requires four bits because

$$|K| = 16 = 2^4.$$

Therefore

$$15 + 4 = 19$$

bits suffice.

Hence the normal form reaches the finite information lower bound:

$$\boxed{\text{program state} = \text{15-bit visible operation} + \text{4-bit silent residual}}$$

The residual memory is silent only relative to the original seed. Under founder application, the four-bit state generally changes.

For founder g_i, define the correction mask

$$c_i(f) \in \{0,1\}^4$$

by

$$g_i \circ s(f) = s(g_i[f]) \circ c_i(f).$$

Then the normal-form update rule is

$$(f, \varepsilon) \longmapsto gi(g_i[f], \varepsilon \oplus c_i(f)).$$

Thus the visible operation evolves by table transformation, while the silent memory evolves by a four-bit correction.

The complete correction system contains

$$5 \times 19{,}683 = 98{,}415$$

masks.

It converts the large reversible program ecology into an exact finite-state rewriting system:

visible table + four residual bits + founder input $\longrightarrow$ new visible table + new residual bits.

10. Frontier-Birth Certification

The canonical section

$$s: \mathcal{O} \to \mathcal{E}$$

was first obtained by breadth-first exploration of the five-founder transition graph.

A direct storage of all canonical contexts requires one context code for each operation:

$$19{,}683 \times 4 = 78{,}732$$

bytes.

The complete founder-correction memory requires

$$5 \times 19{,}683 = 98{,}415$$

additional bytes.

Together these two memories occupy

$$78{,}732 + 98{,}415 = 177{,}147$$

bytes.

The system replaces both as primary knowledge with a compact record of first births.

Let the operations be processed in deterministic breadth-first order. For each processed operation f, define the five-bit birth mask

$$\beta(f) = (\beta_0(f), \dots, \beta_4(f)),$$

where

$$\beta_i(f) = 1$$

exactly when the transition

$$f \longrightarrow gig_i[f]$$

creates an operation not previously encountered.

Otherwise,

$$\beta_i(f) = 0.$$

Thus each processed state records only which founder applications enlarge the known frontier.

The complete birth certificate contains

$$19{,}683$$

five-bit masks. Its logical size is

$$19{,}683 \times 5 = 98{,}415$$

bits.

Packed into bytes, this requires

$$\left\lceil \frac{98{,}415}{8} \right\rceil = 12{,}302$$

bytes.

Proposition 11. Frontier reconstruction

The ordered birth masks, together with the seed and fixed founder order, reconstruct the complete breadth-first canonical section.

Proof

Begin with the queue containing only the seed w.

Process queued operations in order. For the current operation f, inspect its five-bit mask. Whenever $\beta_i(f) = 1$, create the child

$$h = g_i[f],$$

assign f as its canonical parent, assign g_i as its canonical founder, and append h to the queue.

Because every nonseed operation is born exactly once, the procedure reconstructs all 19,682 parent edges. The recovered parent tree is identical to the original deterministic breadth-first construction. Therefore the same canonical section is obtained. ■

The certificate contains exactly

$$19{,}682$$

first-birth edges, one for every nonseed operation.

The reconstructed parent relation yields all canonical founder words and reproduces:

$$\max_{f \in \mathcal{O}} |\, s(f) \,| = 9,$$
$$\sum_{f \in \mathcal{O}} |\, s(f) \,| = 137{,}885.$$

Once the canonical section has been reconstructed, every correction mask is compiled by

$$c_i(f) = \rho_S(s(g_i[f])^{-1} \circ g_i \circ s(f)).$$

Thus the correction table is not primitive memory. It is a derived consequence of:

$$\text{seed} + \text{founders} + \text{birth certificate} + \text{residual coordinates}.$$

Exact regeneration recovers both former binary memories byte-for-byte.

The storage reduction is

$$177{,}147 - 12{,}302 = 164{,}845$$

bytes.

The fractional reduction is

$$\frac{164{,}845}{177{,}147} \approx 0.9306.$$

Hence

$$93.06\%$$

of the previous active binary memory is eliminated.

The expansion ratio is

$$\frac{177{,}147}{12{,}302} \approx 14.40.$$

Therefore one byte of frontier memory regenerates approximately 14.4 bytes of canonical and rewriting memory.

All $2^5 = 32$possible birth masks occur. Consequently, five bits per state are necessary for any direct fixed-width encoding of this mask alphabet.

This establishes only a scoped encoding bound. It does not prove that the complete developmental history is globally incompressible.

The frontier certificate changes the epistemic status of the rewrite system:

$$\text{stored answers} \longrightarrow \text{stored developmental history} \longrightarrow \text{compiled answers}.$$

The canonical transversal and correction table become executable consequences rather than primary data.

The final 973 masks contain no first births. Thus all 19,683 operations have appeared after processing only

$$18{,}710$$

frontier states.

The productive prefix therefore has logical size

$$18{,}710 \times 5 = 93{,}550$$

bits and packed size

$$11{,}694$$

bytes.

This productive birth history is sufficient for complete reconstruction of the operation universe and forms the basis of random-access navigation.

11. Multiscale Random-Access Navigation

The productive frontier certificate permits complete reconstruction, but direct replay remains inefficient for isolated queries.

To recover one canonical program by sequential replay, an earlier streaming navigator processed birth masks until the target operation first appeared. Median canonicalization required

$$3{,}613$$

mask reads, with worst-case termination at mask

$$18{,}710.$$

The multiscale navigator replaces repeated frontier replay with three coordinated indices:

operation code → birth rank → local birth block → canonical parent chain.

For each operation $f \in \mathcal{O}$, let

$$r(f) \in \{0, \dots, 19{,}682\}$$

denote its position in deterministic breadth-first birth order.

Because

$$19{,}683 < 2^{15},$$

each birth rank is represented by fifteen bits.

The complete rank permutation therefore occupies

$$19{,}683 \times 15 = 295{,}245$$

bits, or

$$36{,}906$$

packed bytes.

The productive birth certificate is partitioned into blocks of 64 masks.

For block j, store the cumulative number

$$B_j$$

of operations born before that block.

There are

$$\left\lceil \frac{18{,}710}{64} \right\rceil = 293$$

productive blocks, together with the initial checkpoint, giving 294 cumulative values.

The checkpoint memory occupies

$$552$$

bytes.

The complete navigation index is therefore

$$11{,}694 + 36{,}906 + 552 = 49{,}152$$

bytes.

Since

$$49{,}152 = 48 \times 1{,}024,$$

the navigator occupies exactly 48 KiB.

The block width 64 is selected by the frozen rule:

Choose the finest power-of-two checkpoint spacing for which the complete index does not exceed 48 KiB.

A width of 32 would require 49,699 bytes and therefore violate the declared budget.

Canonicalization algorithm

Given a target operation f:

1. read its birth rank $r(f)$;

2. locate the unique 64-mask block containing that rank;
3. use the cumulative checkpoint to establish the block's initial birth count;
4. scan only the local masks until the birth edge of f is found;
5. recover its canonical parent $p(f)$ and founder label $i(f)$;
6. repeat for $p(f)$;
7. terminate at the original seed w.

Since canonical depth is at most nine, parent recovery requires at most nine iterations.

Reversing the recovered founder labels produces the exact shortest program

$$s(f).$$

Theorem 12. Exact indexed canonicalization

For every operation $f \in \mathcal{O}$, the multiscale navigator reconstructs the same canonical section selected by the original deterministic breadth-first search.

Proof

The birth-rank permutation identifies the exact first-discovery position of f. The checkpoint and local mask scan recover the unique certified first-birth edge at that position. This yields the same canonical parent and founder label as the original breadth-first construction.

Repeated recovery follows the certified parent tree to the seed. Therefore the reconstructed founder word is identical to the frozen canonical word and has minimum length. ∎

Across all 19,683 operations, the number of local mask reads is:

$$\begin{aligned}
&\text{median} = 153,\\
&P_{90} = 215,\\
&P_{99} = 262,\\
&\max = 344.
\end{aligned}$$

The worst indexed query occurs for operation code

$$4969.$$

Its shortest founder word is

$$g_0 g_3 g_1 g_2 g_1 g_4 g_1 g_3 g_0,$$

of length nine.

The indexed method therefore replaces a median sequential scan of 3,613 masks with a median local inspection of 153 masks.

The approximate reduction is

$$1 - \frac{153}{3{,}613} \approx 95.8\%.$$

Direct correction recovery

For founder $g_i$and source operation f, the correction mask is

$$c_i(f) = \rho_S(s(g_i[f])^{-1} \circ g_i \circ s(f)).$$

The navigator therefore performs two canonicalization queries:

$$s(f), s(g_i[f]).$$

It stores neither canonical context permanently nor any correction entry.

Across all

$$5 \times 19{,}683 = 98{,}415$$

founder-source pairs, correction recovery requires:

$$\text{median} = 309$$

local mask reads,

$$\begin{aligned} P_{90} &= 407, \\ P_{99} &= 482, \\ \max &= 625. \end{aligned}$$

Every derived correction reproduces the frozen rewrite memory exactly.

Hence every normal-form transition

$$(f, \varepsilon) \longmapsto gi(g_i[f], \varepsilon \oplus c_i(f))$$

is answered without storing either the 19,683 canonical programs or the 98,415 correction masks.

The complete verification covers

$$5 \times 19{,}683 \times 16 = 1{,}574{,}640$$

program-state transitions.

All are exact.

The memory progression is therefore

$$177{,}147 \text{ bytes}$$

for direct canonical and correction memories,

$$55{,}640 \text{ bytes}$$

for streaming frontier navigation, and

$$49{,}152 \text{ bytes}$$

for multiscale direct navigation.

The final architecture is

$$\boxed{\begin{array}{c}\text{seed} + \text{productive birth history} + \text{birth-rank permutation} + \text{sparse checkpoints} \to \\ \text{exact random-access shortest programs}\end{array}}$$

The system no longer stores the mathematical answers it returns. It stores a compact index into the developmental process by which those answers first appeared.

12. Mathematical Significance and Limitations

The construction begins with one finite algebra

$$\mathcal{S}_0 = (A, w), A = \{0,1,2\},$$

where

$$w = 201/122/010.$$

From this seed, the system generates:

all $19{,}683$ binary operations on A,
$314{,}928$ reversible context programs,
16 program representatives per operation,
1 canonical shortest representative per operation,
4 residual bits per program,

and

1 exact 48-KiB random-access navigator.

The principal developmental sequence is

$\boxed{\text{operation} \to \text{terms} \to \text{complete function universe} \to \text{reversible contexts} \to \text{connected program ecology} \to \text{quo}}$

The result is not merely an enumeration of finite tables. Each operation receives:

1. a term witness over the original seed;
2. a reversible path from the seed;
3. a shortest selected founder program;
4. a four-bit residual coordinate;
5. an exact indexed reconstruction procedure.

The system therefore converts a finite function space into an internally generated computational world.

12.1 Generation from minimal primitives

The invention layer is restricted to:

variables,
w,
substitution,

composition,
complete evaluation,

and

extensional equality.

Named structures such as groups, semigroups, lattices, kernels, ranks, and congruences are not supplied during invention.

When recognizable algebraic structure appears, it is assigned only after exact computation.

For example, the residual memory is discovered operationally as:

16 states,
15 nonidentity self-inverse transformations,
pairwise commutation,
4 minimum generators.

Only afterward is it identified with

$$(\mathbb{Z}/2\mathbb{Z})^4.$$

This separation distinguishes machine-generated structure from human interpretation.

12.2 Exactness

All claims concern finite sets and are exhaustively decidable.

Term equality is checked on all inputs.

Context reversibility is checked at all 27 ternary assignments.

Operation-universe coverage is checked across all 19,683 tables.

Program closure is checked across all generated contexts.

Canonical shortest paths are certified by deterministic breadth-first search.

Rewrite correctness is checked across all

$$1{,}574{,}640$$

normal-form transitions.

Thus no statistical inference is required for the principal theorems.

The epistemic form is

$$\text{claim} + \text{complete finite certificate} + \text{independent rebuild.}$$

12.3 Relation to functional completeness

The seed satisfies

$$\mathrm{Clo}_2(w) = A^{A^2}.$$

Hence every binary operation on the three-element carrier is term-definable from w.

This resembles functional completeness in many-valued logic, but the present construction does not begin with designated truth values or logical connectives.

The table is treated anonymously.

Logical interpretation is therefore optional and posterior.

The result is more accurately stated as finite clone completeness:

$$w \text{ generates the full binary operation clone on } A.$$

12.4 Representation as mathematics

After functional saturation, new terms cease to create new binary functions. They instead create alternative programs for known functions.

The developmental resource therefore changes from function novelty to representation multiplicity:

$$\begin{aligned}&\text{before saturation:new term} \Rightarrow \text{possibly new operation,}\\&\text{after saturation:new term} \Rightarrow \text{new representation of an old operation.}\end{aligned}$$

The sixteenfold program fibers are not treated as waste. They become the residual memory K.

Thus redundancy produces structure:

$$\text{multiple names} \rightarrow \text{equivalence classes} \rightarrow \text{residual coordinates} \rightarrow \text{rewriting dynamics.}$$

12.5 Canonicalization

The canonical section

$$s: \mathcal{O} \to \mathcal{E}$$

selects one shortest founder program per visible operation.

This is a computational normal form.

It converts the quotient map

$$\pi: \mathcal{E} \to \mathcal{O}$$

into the factorization

$$p = s(\pi(p))\, r(p).$$

The visible component records the operation. The residual component records the choice of program name.

The construction is analogous to choosing representatives of cosets, although the section is determined by breadth-first discovery rather than by an externally supplied algebraic convention.

12.6 Developmental memory

The frontier certificate stores not the final answers but the order in which new states first appeared.

From this history, the system regenerates:

canonical parents,
shortest programs,
residual corrections,

and

complete rewrite behavior.

This suggests a general principle:

$$\boxed{\text{a compact developmental history may generate a much larger mathematical memory}}$$

The 12,302-byte birth certificate regenerates 177,147 bytes of former direct memory.

The 48-KiB index then supports random access without reconstructing the entire history for each query.

12.7 Limitations

The present system remains finite and highly specific.

First, the carrier is fixed:

$$|A| = 3.$$

Second, the principal universe concerns binary operations:

$$A^2 \to A.$$

Third, the developmental root is one selected table:

$$201/122/010.$$

Fourth, the five-founder system is not claimed to be unique.

Fifth, the canonical section depends on the frozen founder order.

Sixth, recurrence, reversibility, shortest path, and memory compression remain generic human-selected objectives.

Seventh, the 48-KiB navigator is not proven globally optimal.

Eighth, successful finite closure does not establish scalability to larger carriers, higher arities, or infinite structures.

The construction therefore does not yet imply autonomous discovery of general mathematics.

It establishes a finite proof of principle:

minimal seed ⇒ complete generated universe ⇒ internally organized mathematical memory.

12.8 Open developmental questions

The next questions are structural.

Can the birth-rank permutation be compressed while retaining direct lookup?

Can the system invent its own criteria for choosing useful contexts?

Can analogous founder ecologies arise on larger carriers?

Can higher-arity operations be generated and indexed from the same seed?

Can internally discovered quotients produce numbers, orders, geometries, or topologies without those concepts being supplied?

Can canonical program memory support conjecture formation across independently generated worlds?

The long-term problem is not merely to enumerate more finite operations.

It is to determine whether the sequence

$$\text{seed} \to \text{representation} \to \text{equivalence} \to \text{compression} \to \text{theory}$$

can continue without importing the target theory in advance.

12.9 Conclusion

One anonymous 3×3table generates every binary operation on a three-element set.

Seed-derived contexts connect the complete operation universe reversibly.

The resulting program redundancy factors into a visible operation and a four-bit silent residual.

A canonical section assigns every operation a shortest founder program of length at most nine.

A compact developmental certificate and multiscale index recover those programs and their rewrite corrections exactly.

The final result is

$$\boxed{201/122/010 \longrightarrow 19{,}683 \text{ operations} \longrightarrow 314{,}928 \text{ programs} \longrightarrow 48\text{-KiB exact navigation}}$$

The construction demonstrates that a single finite operation can serve simultaneously as generator, transformation language, memory substrate, and navigation root for a complete mathematical micro-universe.

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