Graphics of the Platonic Solids in HoTT and Univalent Foundations

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This short note accompanies a downloadable 12 × 18-inch poster presenting the five Platonic solids through the combined perspectives of polyhedral geometry, topology, incidence theory, symmetry, and homotopy type theory. The tetrahedron, cube, octahedron, dodecahedron, and icosahedron are distinct regular polyhedra: their vertex sets, edge graphs, face structures, metric realizations, and symmetry data are not interchangeable. Nevertheless, each boundary is a regular cell decomposition of the 2-sphere and therefore has the same homeomorphism and homotopy type. Polyhedral duality exchanges vertices with faces and identifies the boundary-face poset of a dual polyhedron with the opposite of the original face poset. The nerve and geometric realization of the proper boundary-face poset recover the barycentric subdivision of the boundary. Rotational symmetry produces the classical spherical orbifolds with signatures $(2,3,3)$, $(2,3,4)$, and $(2,3,5)$. Univalence then clarifies the precise level at which equivalence may be treated as identity: equivalent boundary types may be identified in a univalent universe, whereas the five complete structured polyhedra remain distinct. The poster is intended as a compact visual atlas of these relationships.

Keywords: Platonic solids, homotopy type theory, HoTT, univalence, polyhedral duality, face posets, barycentric subdivision, spherical orbifolds, regular polyhedra, symmetry groups, 2-sphere, downloadable poster.

1. Purpose

The accompanying poster is a visual summary of the structural relationships among the five Platonic solids. Its central claim is not that the five polyhedra are one object without qualification. Rather, identity depends on the structure retained by the chosen mathematical universe.

At the level of regular polyhedra, the five solids are distinct. At the level of boundary topology, every Platonic solid presents the same spherical type. At the level of incidence, dual pairs are related by order reversal. At the level of rotational symmetry, the solids fall into three symmetry classes. At the level of homotopy type theory, genuine equivalences between types may be converted into identities by univalence.

The poster therefore separates several notions that are often conflated:

geometric equality,combinatorial isomorphism,duality,homeomorphism,homotopy equivalence,univalent ident

2. The Five Regular Polyhedra

A Platonic solid is a convex polyhedron whose faces are congruent regular p-gons and for which exactly $q$faces meet at every vertex. It is denoted by the Schläfli symbol

$$\{p, q\}.$$

The five possibilities are

$$\begin{aligned} T &= \{3,3\}, \\ C &= \{4,3\}, \\ O &= \{3,4\}, \\ D &= \{5,3\}, \\ I &= \{3,5\}. \end{aligned}$$

Their vertex-edge-face counts are

P	V	E	F
T	4	6	4
C	8	12	6
O	6	12	8
D	20	30	12
I	12	30	20

For the boundary complex of each solid,

$$\boxed{\chi(\partial P) = V - E + F = 2.}$$

The equality is the Euler characteristic of the polyhedral boundary, not of the filled three-dimensional solid. The filled solid is homeomorphic to a closed 3-ball and has Euler characteristic 1.

Proposition 2.1. The five Platonic solids are pairwise distinct as regular polyhedra.

Proof. Their Schläfli symbols and vertex-edge-face data differ. Hence their complete regular cell structures cannot be mutually isomorphic. ■

3. Shared Boundary Homotopy Type

For every Platonic solid P, the boundary $\partial P$is a closed connected 2-manifold embedded in $\mathbb{R}^3$. Convexity implies

$$\partial P \cong S^2.$$

Consequently,

$$\boxed{\partial P \simeq S^2.}$$

Thus all five boundaries have the same homotopy type even though they carry different regular cell decompositions.

Proposition 3.1. For any two Platonic solids $P$and Q,

$$\partial P \cong \partial Q$$

as topological spaces.

Proof. Both spaces are homeomorphic to S^2. ■

The conclusion does not imply that $P$and $Q$are the same polyhedron. The homeomorphism forgets metric data, face shape, incidence labels, and the particular decomposition of the sphere.

4. Duality as Opposite Incidence

Let P^*denote the dual of P. The Schläfli symbol transforms by

$$\{p, q\}^* = \{q, p\}.$$

Hence

$$\begin{gathered} C^* = O, O^* = C, \\ D^* = I, I^* = D, \end{gathered}$$

and

$$T^* \cong T.$$

Let $\mathrm{Face}_\partial(P)$denote the poset of nonempty proper faces of the boundary complex, ordered by inclusion. Duality reverses containment and gives

$$\boxed{\mathrm{Face}_\partial(P^*) \cong \mathrm{Face}_\partial(P)^{\mathrm{op}}.}$$

Vertices of Pcorrespond to faces of P^*, edges correspond to edges, and faces correspond to vertices.

Duality is therefore not an ordinary continuous deformation from one regular polyhedron into another. It is a contravariant equivalence of incidence structures.

5. Face Posets, Nerves, and Realization

The combinatorial boundary of a polyhedron may be recovered from its face relations.

Let

$$N\,(\mathrm{Face}_\partial(P))$$

be the nerve of the boundary-face poset. Its simplices are chains

$$\sigma_0 < \sigma_1 < \cdots < \sigma_k$$

of incident faces.

The geometric realization of this nerve is the order complex of the face poset. It is naturally identified with the barycentric subdivision of the boundary:

$$\boxed{|\,N\,(\mathrm{Face}_\partial(P))\,| \cong |\,\mathrm{sd}(\partial P)\,| \cong S^2.}$$

The vertices of the subdivision are barycenters of the original vertices, edges, and faces. Incidence chains become simplices.

Theorem 5.1. The boundary incidence data of a Platonic solid determine a simplicial realization homeomorphic to S^2.

Proof. The order complex of the proper face poset is the barycentric subdivision of the boundary complex. Barycentric subdivision preserves geometric realization and homeomorphism type. ■

The edge graph alone is insufficient for this reconstruction without additional incidence information. The full boundary-face poset records the required dimensions and containments.

6. Symmetry Groups and Spherical Orbifolds

The orientation-preserving symmetry groups of the Platonic solids are

$$\mathrm{Rot}(T) \cong A_4,$$
$$\mathrm{Rot}(C) \cong \mathrm{Rot}(O) \cong S_4,$$
$$\mathrm{Rot}(D) \cong \mathrm{Rot}(I) \cong A_5.$$

These groups have orders

$$|A_4| = 12, |S_4| = 24, |A_5| = 60.$$

The action of a rotational group G on the spherical boundary gives a quotient orbifold

$$S^2/G.$$

The corresponding signatures are

$$(2, 3, 3)$$

for tetrahedral symmetry,

$$(2, 3, 4)$$

for cubic-octahedral symmetry, and

$$(2, 3, 5)$$

for dodecahedral-icosahedral symmetry.

For a regular polyhedron $\{p, q\}$, this may be written, up to permutation, as

$$\boxed{S^2/G(p,q) \text{ has signature } (2,p,q).}$$

Dual polyhedra share the same rotational group and the same spherical orbifold because duality preserves the underlying symmetry action while exchanging face and vertex roles.

7. Univalence and Level-Dependent Identity

Let $\mathcal{U}$be a univalent universe and let $A, B: \mathcal{U}$. The univalence principle supplies a canonical map

$$\boxed{\mathrm{ua}_{A,B}: (A \simeq B) \longrightarrow (A =_{\mathcal{U}} B).}$$

Thus an equivalence between types determines an identity path in the universe.

Applied to the Platonic boundaries, an equivalence

$$e: \partial P \simeq \partial Q$$

determines

$$\mathrm{ua}(e): \partial P =_{\mathcal{U}} \partial Q.$$

This statement concerns the boundary types regarded as objects of $\mathcal{U}$. It does not imply

$$P = Q$$

as regular metric polyhedra, incidence structures, embedded subsets of $\mathbb{R}^3$, or symmetry-equipped cell complexes.

Principle 7.1. Univalent identity is indexed by the structure encoded in the type.

If a type remembers only the homotopy type of the boundary, all five Platonic boundaries may be identified. If it remembers the complete regular cell decomposition, the distinctions remain. If it remembers duality only up to opposite incidence, dual pairs are related but not collapsed into literal equality without an explicitly chosen equivalence of the relevant structured types.

Univalence therefore does not indiscriminately erase structure. It makes the level of retained structure mathematically explicit.

8. Structural Synthesis

The five Platonic solids exhibit a hierarchy of relationships:

distinct regular polyhedra, homeomorphic spherical boundaries, dual opposite-incidence pairs, equivalent nerve realizations, three rotational symmetry classes, univalent identities only at equivalent structural levels.

The common object is not a single Platonic polyhedron. The common object is the spherical boundary type after the distinctions encoded by the regular cell structure have been forgotten.

The poster visualizes this controlled passage from distinct geometric realizations to shared topological and homotopical structure. Its governing claim is therefore:

$$\boxed{\text{One boundary type, many cell structures, and identity only under specified equivalence.}}$$

9. Poster Specifications

The accompanying poster is formatted for large-scale printing with the following specifications:

$$12 \times 18 \text{ inches},$$

portrait orientation, full color, and 300 dpi.

10. References

1. Coxeter, H. S. M. *Regular Polytopes*. Dover Publications, 1973.
2. Grünbaum, B. *Convex Polytopes*. Springer, 2003.
3. Ziegler, G. M. *Lectures on Polytopes*. Springer, 1995.
4. Björner, A. “Topological Methods.” In *Handbook of Combinatorics*. Elsevier, 1995.
5. Thurston, W. P. *The Geometry and Topology of Three-Manifolds*. Princeton University lecture notes, 1979.
6. The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study, 2013.
7. Awodey, S. “Type Theory and Homotopy.” *Epistemology versus Ontology*, Springer, 2012.
8. Kapranov, M., and Voevodsky, V. “∞-Groupoids and Homotopy Types.” *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 1991.

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