**Phase-Driven Tuple Imagery: Tensor Factorization of a Million-Intersection RGB Field**

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Tuple Imagery is a finite mathematical construction in which cyclic scalar samples are lifted to ordered color triples and combined through componentwise addition subject to a unit-cube veto. For ten phase samples, the construction produces $10^3 = 1000$ indexed RGB sources and $10^6$ ordered pairwise intersections. Despite this apparent dimensional expansion, the admissibility structure of the complete image is determined exactly by a single $10 \times 10$binary matrix. If $M_\theta$records which one-dimensional sums lie in $[0, 1]$, then the full six-index admissibility tensor is

$$\mathrm{supp}_{\mathrm{adm}}(I_\theta) = M_\theta \otimes M_\theta \otimes M_\theta.$$

Consequently, if $m_\theta$ entries of $M_\theta$ are admissible, then exactly $m_\theta^3$RGB intersections survive the veto. For the historical ten-sample sine generator, antipodal phase symmetry implies $u_{j+5} = -u_j$. At generic phases this yields $m_\theta = 55$, hence $166{,}375$ admissible intersections. The construction therefore transforms a one-dimensional cyclic waveform into a rigorously factorized, phase-evolving million-cell color field.

**Keywords:** tuple imagery; tensor product; cyclic sampling; RGB field; admissibility mask; phase dynamics; finite visualization; componentwise addition

Video: https://www.youtube.com/watch?v=bwnY4gByhXQ

Google Drive:
https://drive.google.com/drive/folders/15Kd8bUjm7zVaKdVIwbPhqjjV68aUZQvc?usp=sharing

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**1. Cyclic Scalar Source**

Let

$$\mathbb{Z}_{10} = \{0,1,\dots,9\}$$

with indices interpreted modulo $10$.

**Definition 1. Phase source**

For phase $\theta \in \mathbb{R}$, define

$$u_j(\theta) = b + a\sin\left(\theta + \frac{2\pi j}{10}\right) + h\sin\left(2\theta + \frac{4\pi j}{10}\right), j \in \mathbb{Z}_{10},$$

where $a$is the fundamental amplitude, $b$is the vertical offset, and $h$is the second-harmonic amplitude.

The historical Tuple Imagery source is the specialization

$$a = \frac{1}{2}, b = 0, h = 0,$$

so that

$$u_j(\theta) = \frac{1}{2}\sin\left(\theta + \frac{2\pi j}{10}\right). \qquad (1)$$

The source vector is

$$U_\theta = (u_0(\theta), u_1(\theta), \dots, u_9(\theta)) \in \mathbb{R}^{10}.$$

**Proposition 1. Cyclic phase transport**

For every $j \in \mathbb{Z}_{10}$,

$$u_j\left(\theta + \frac{2\pi}{10}\right) = u_{j+1}(\theta).$$

**Proof**

Direct substitution gives

$$u_j\left(\theta+\frac{2\pi}{10}\right)=\frac{1}{2}\sin\left(\theta+\frac{2\pi(j+1)}{10}\right)=u_{j+1}(\theta).$$

Thus continuous phase evolution acts as cyclic index transport after each increment $2\pi/10$. ■

**Proposition 2. Antipodal symmetry**

For the historical generator,

$$u_{j+5}(\theta)=-u_j(\theta). \qquad (2)$$

**Proof**

Since

$$\frac{2\pi(j+5)}{10}=\frac{2\pi j}{10}+\pi,$$

equation (1) gives

$$u_{j+5}(\theta)=\frac{1}{2}\sin\left(\theta+\frac{2\pi j}{10}+\pi\right)=-\frac{1}{2}\sin\left(\theta+\frac{2\pi j}{10}\right).$$

Therefore $u_{j+5}=-u_j$. ■

Equation (2) is the central exact symmetry of the historical construction. The ten samples form five antipodal scalar pairs.

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**2. Tuple Lift**

Let

$$A=\mathbb{Z}_{10}^3.$$

An element

$$\alpha=(\alpha_1,\alpha_2,\alpha_3)\in A$$

indexes one ordered RGB source.

**Definition 2. Tuple source**

For each $\alpha \in A$, define

$$G_\theta(\alpha) = (u_{\alpha_1}(\theta), u_{\alpha_2}(\theta), u_{\alpha_3}(\theta)) \in \mathbb{R}^3. \qquad (3)$$

Because

$$|A| = 10^3 = 1000,$$

the scalar source $U_\theta$generates $1000$indexed triples.

These triples are indexed objects. Distinct indices remain distinct even when two numerical triples happen to coincide.

**Definition 3. Additive interaction field**

For $\alpha, \beta \in A$, define the unrestricted interaction

$$F_\theta(\alpha, \beta) = G_\theta(\alpha) + G_\theta(\beta), \qquad (4)$$

where addition is componentwise.

Thus

$$F_\theta(\alpha, \beta) = \left(u_{\alpha_1} + u_{\beta_1}, u_{\alpha_2} + u_{\beta_2}, u_{\alpha_3} + u_{\beta_3}\right).$$

The domain has cardinality

$$|A \times A| = 10^3 \cdot 10^3 = 10^6.$$

Hence the full Tuple Imagery field consists of one million ordered interactions.

**Proposition 3. Coordinate separability**

Each coordinate of $F_\theta(\alpha, \beta)$ depends only on the corresponding pair of scalar indices:

$$(F_\theta(\alpha, \beta))_r = u_{\alpha_r}(\theta) + u_{\beta_r}(\theta), r = 1{,}2{,}3. \qquad (5)$$

This elementary separability is responsible for the exact tensor factorization of the complete field.

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### 3. Veto Map and Admissibility

Let

$$C = [0,1]^3$$

denote the closed RGB unit cube.

**Definition 4. Hard veto**

Define

$$V: \mathbb{R}^3 \longrightarrow C$$

by

$$V(x) = \begin{cases} x, & x \in C, \\ (0,0,0), & x \notin C. \end{cases} \qquad (6)$$

The rendered Tuple Imagery field is

$$I_\theta(\alpha, \beta) = V(F_\theta(\alpha, \beta)). \qquad (7)$$

An interaction is **admissible** when

$$F_\theta(\alpha, \beta) \in [0,1]^3.$$

The admissible support is therefore

$$\Sigma_\theta = \{(\alpha, \beta) \in A^2 : F_\theta(\alpha, \beta) \in [0,1]^3\}. \qquad (8)$$

This must be distinguished from literal nonblack support. An admissible interaction whose sum is exactly $(0, 0, 0)$ is encoded as black even though it lies within the cube.

**Definition 5. One-channel mask**

Define

$$M_\theta : \mathbb{Z}_{10}^2 \longrightarrow \{0,1\}$$

by

$$M_\theta(i,j) = \mathbf{1}_{[0,1]}(u_i(\theta) + u_j(\theta)). \qquad (9)$$

Let

$$m_\theta = \sum_{i,j \in \mathbb{Z}_{10}} M_\theta\,(i,j) \qquad (10)$$

be the number of admissible ordered scalar pairs.

**Theorem 1. Tensor-factorization theorem**

For every $\alpha, \beta \in A$,

$$\mathbf{1}_{\Sigma_\theta}(\alpha,\beta) = \prod_{r=1}^{3} M_\theta\,(\alpha_r, \beta_r). \qquad (11)$$

Equivalently, after lexicographic indexing,

$$\mathbf{1}_{\Sigma_\theta} = M_\theta \otimes M_\theta \otimes M_\theta. \qquad (12)$$

**Proof**

By equation (5),

$$F_\theta(\alpha,\beta) \in [0,1]^3$$

if and only if

$$u_{\alpha_r}(\theta) + u_{\beta_r}(\theta) \in [0,1]$$

for each $r \in \{1,2,3\}$. Each coordinate condition is represented by

$$M_\theta(\alpha_r, \beta_r) = 1.$$

All three conditions hold exactly when their product equals one. This proves equation (11). Equation (12) is its matrix representation under product indexing. ■

**Corollary 1. Exact admissible count**

The number of admissible RGB interactions is

$$|\Sigma_\theta| = m_\theta^3. \qquad (13)$$

**Proof**

Each RGB coordinate independently selects one of the $m_\theta$ admissible scalar pairs. Therefore the three-coordinate product contains

$$m_\theta \cdot m_\theta \cdot m_\theta = m_\theta^3$$

admissible interactions. ■

**4. Exact Geometry of Scalar Intersections**

For the historical generator, define the scalar interaction

$$s_{ij}(\theta) = u_i(\theta) + u_j(\theta). \qquad (14)$$

Using the sum identity for sine,

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right),$$

equation (1) gives

$$s_{ij}(\theta) = \sin\left(\theta + \frac{\pi(i+j)}{10}\right)\cos\left(\frac{\pi(i-j)}{10}\right). \qquad (15)$$

Equation (15) separates each interaction into a **phase factor**

$$\sin\left(\theta+\frac{\pi(i+j)}{10}\right)$$

and an **index-separation factor**

$$\cos\left(\frac{\pi(i-j)}{10}\right).$$

Since

$$-\frac{1}{2}\leq u_j(\theta)\leq\frac{1}{2},$$

every scalar sum satisfies

$$-1\leq s_{ij}(\theta)\leq 1.$$

Therefore the upper veto condition is automatic:

$$s_{ij}(\theta)\leq 1.$$

The scalar interaction is admissible precisely when

$$s_{ij}(\theta)\geq 0. \qquad (16)$$

Thus

$$M_\theta(i,j)=\mathbf{1}_{[0,\infty)}\left[\sin\left(\theta+\frac{\pi(i+j)}{10}\right)\cos\left(\frac{\pi(i-j)}{10}\right)\right]. \qquad (17)$$

**Definition 6. Zero-sum multiplicity**

Let

$$Z_\theta=\left|\left\{(i,j)\in\mathbb{Z}_{10}^2: s_{ij}(\theta)=0\right\}\right|. \qquad (18)$$

**Proposition 4. Permanent zero interactions**

For every $\theta$,

$$s_{i,i+5}(\theta) = 0. \tag{19}$$

Hence

$$Z_\theta \geq 10.$$

**Proof**

By antipodal symmetry,

$$u_{i+5}(\theta) = -u_i(\theta).$$

Therefore

$$s_{i,i+5}(\theta) = u_i(\theta) + u_{i+5}(\theta) = 0.$$

There are ten ordered pairs of the form $(i, i+5)$. ■

These permanent zero interactions form the cyclic antidiagonal

$$\Delta^- = \{(i,j) \in \mathbb{Z}_{10}^2 : j - i \equiv 5(\mathrm{mod} 10)\}. \tag{20}$$

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## 5. Sign-Pairing and the Exact Count

Define the involution

$$\tau: \mathbb{Z}_{10}^2 \longrightarrow \mathbb{Z}_{10}^2$$

by

$$\tau(i,j) = (i+5, j+5). \tag{21}$$

**Proposition 5. Sign-reversing involution**

For every ordered pair $(i, j)$,

$$s_{\tau(i,j)}(\theta) = -s_{ij}(\theta). \qquad (22)$$

**Proof**

Using equation (2),

$$\begin{aligned} s_{i+5,j+5}(\theta) &= u_{i+5}(\theta) + u_{j+5}(\theta) \\ &= -u_i(\theta) - u_j(\theta) \\ &= -s_{ij}(\theta). \end{aligned}$$

■

The involution $\tau$ has no fixed ordered pair. Consequently, every positive interaction is paired with a negative interaction, while zero interactions are paired among themselves.

**Theorem 2. Scalar admissibility law**

For every phase $\theta$,

$$m_\theta = \frac{100 + Z_\theta}{2}. \qquad (23)$$

**Proof**

Let $P_\theta$, $N_\theta$, and $Z_\theta$ denote the numbers of positive, negative, and zero scalar sums. Then

$$P_\theta + N_\theta + Z_\theta = 100.$$

Proposition 5 gives

$$P_\theta = N_\theta.$$

Hence

$$2P_\theta + Z_\theta = 100,$$

so

$$P_\theta = \frac{100 - Z_\theta}{2}.$$

Admissibility includes positive and zero sums. Therefore

$$m_\theta = P_\theta + Z_\theta = \frac{100 - Z_\theta}{2} + Z_\theta = \frac{100 + Z_\theta}{2}.$$

■

Thus the complete three-dimensional admissibility count is controlled entirely by the number of exact scalar cancellations:

$$|\Sigma_\theta| = \left(\frac{100 + Z_\theta}{2}\right)^3. \qquad (24)$$

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**6. Generic and Singular Phases**

Equation (15) shows that a scalar interaction vanishes when either

$$\cos\left(\frac{\pi(i-j)}{10}\right) = 0 \qquad (25)$$

or

$$\sin\left(\theta + \frac{\pi(i+j)}{10}\right) = 0. \qquad (26)$$

Condition (25) is equivalent to

$$i - j \equiv 5 (\mathrm{mod} 10),$$

and therefore gives the ten permanent antipodal cancellations.

For a non-antipodal pair, equation (26) is equivalent to

$$\theta \equiv -\frac{\pi(i+j)}{10} (\mathrm{mod} \pi). \qquad (27)$$

Hence additional cancellations occur only on the phase lattice

$$\mathcal{E} = \left\{\frac{k\pi}{10} : k \in \mathbb{Z}\right\}. \qquad (28)$$

Modulo $2\pi$, there are twenty singular phases:

$$\mathcal{E}_{2\pi} = \left\{0, \frac{\pi}{10}, \frac{2\pi}{10}, \dots, \frac{19\pi}{10}\right\}. \qquad (29)$$

**Definition 7. Generic phase**

A phase is generic when

$$\theta \notin \mathcal{E}.$$

**Theorem 3. Generic interaction count**

At every generic phase,

$$Z_\theta = 10, m_\theta = 55,$$

and therefore

$$|\Sigma_\theta| = 55^3 = 166{,}375. \qquad (30)$$

**Proof**

Outside $\mathcal{E}$, equation (26) has no solution. The only zero sums are therefore the ten permanent antipodal pairs. Equation (23) gives

$$m_\theta = \frac{100 + 10}{2} = 55.$$

Corollary 1 then yields

$$|\Sigma_\theta| = 55^3 = 166{,}375.$$

■

The generic field therefore contains the proportions

$$\frac{55}{100} = 0.55$$

of admissible scalar interactions and

$$\left(\frac{55}{100}\right)^3 = 0.166375$$

of admissible RGB interactions.

Thus a one-dimensional admissibility density of $55\%$becomes a three-dimensional density of

$$16.6375\%. \qquad (31)$$

This cubic sparsification is not imposed independently. It arises from tensorization.

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**7. Singular-Phase Multiplicities**

Let

$$\theta = \frac{k\pi}{10}, k \in \mathbb{Z}. \qquad (32)$$

Equation (26) then becomes

$$i + j \equiv -k (\mathrm{mod} 10). \qquad (33)$$

This congruence has exactly ten ordered solutions in $\mathbb{Z}_{10}^2$.

The additional zero set is therefore

$$L_k = \{(i, j) \in \mathbb{Z}_{10}^2 : i + j \equiv -k (\mathrm{mod} 10)\}. \qquad (34)$$

The total zero set is

$$\Delta^- \cup L_k.$$

Its cardinality depends on the intersection

$$\Delta^- \cap L_k.$$

A pair in this intersection satisfies

$$j \equiv i + 5(\mathrm{mod}10)$$

and

$$i + j \equiv -k(\mathrm{mod}10).$$

Substitution gives

$$2i + 5 \equiv -k(\mathrm{mod}10),$$

or

$$2i \equiv -k - 5(\mathrm{mod}10). \qquad (35)$$

**Proposition 6. Parity law for singular phases**

If $k$is even, then

$$| \Delta^- \cap L_k |= 0.$$

If $k$is odd, then

$$| \Delta^- \cap L_k |= 2.$$

**Proof**

The congruence

$$2i \equiv -k - 5(\mathrm{mod}10)$$

has a solution precisely when its right-hand side is even.

When $k$is even, $-k - 5$is odd, so there is no solution.

When $k$is odd, $-k-5$is even. Division by 2reduces the congruence modulo 5, producing one solution modulo 5, hence two solutions modulo 10. ■

**Theorem 4. Singular admissibility counts**

At an even singular phase,

$$\theta = \frac{2r\pi}{10},$$

one has

$$Z_\theta = 20, m_\theta = 60,$$

and

$$|\Sigma_\theta| = 60^3 = 216{,}000. \qquad (36)$$

At an odd singular phase,

$$\theta = \frac{(2r+1)\pi}{10},$$

one has

$$Z_\theta = 18, m_\theta = 59,$$

and

$$|\Sigma_\theta| = 59^3 = 205{,}379. \qquad (37)$$

**Proof**

For even $k$, the two ten-element zero sets are disjoint:

$$Z_\theta = 10 + 10 = 20.$$

For odd $k$, their intersection has cardinality two:

$$Z_\theta = 10 + 10 - 2 = 18.$$

The remaining formulas follow from Theorem 2 and Corollary 1. ■

The admissible count therefore takes exactly three values:

$$|\Sigma_\theta| = \begin{cases} 166{,}375, & \theta \notin \mathcal{E}, \\ 216{,}000, & \theta = \dfrac{k\pi}{10}, k \text{ even}, \\ 205{,}379, & \theta = \dfrac{k\pi}{10}, k \text{ odd}. \end{cases} \qquad (38)$$

The singular increases occur because the unit cube is closed: every threshold interaction with sum zero remains admissible.

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## 8. Admissibility Versus Visible Nonblack Support

A zero RGB vector is mathematically admissible but visually black. Define

$$\Sigma_\theta^* = \{(\alpha, \beta) \in \Sigma_\theta : F_\theta(\alpha, \beta) \neq (0,0,0)\}. \qquad (39)$$

Each coordinate of $F_\theta(\alpha, \beta)$vanishes independently by selecting one of the $Z_\theta$zero-sum scalar pairs. Therefore the number of admissible interactions rendered exactly black is

$$Z_\theta^3. \qquad (40)$$

**Corollary 2. Exact visible-support count**

$$|\Sigma_\theta^*| = m_\theta^3 - Z_\theta^3. \qquad (41)$$

Consequently,

$$|\Sigma_\theta^*| = \begin{cases} 165{,}375, & \theta \notin \mathcal{E}, \\ 208{,}000, & \theta = \dfrac{k\pi}{10}, k \text{ even}, \\ 199{,}547, & \theta = \dfrac{k\pi}{10}, k \text{ odd}. \end{cases} \qquad (42)$$

Thus the mathematical support and the visually nonblack support are related but not identical. The distinction is essential when auditing an implementation from rendered pixels alone.

### 9. Symmetry and Covariance

The construction possesses exact symmetries at three distinct levels: exchange symmetry, coordinate symmetry, and cyclic phase covariance.

**Proposition 7. Exchange symmetry**

For all $\alpha, \beta \in A$,

$$F_\theta(\alpha,\beta) = F_\theta(\beta,\alpha),$$

and therefore

$$I_\theta(\alpha,\beta) = I_\theta(\beta,\alpha). \qquad (43)$$

**Proof**

Componentwise addition in $\mathbb{R}^3$ is commutative:

$$G_\theta(\alpha) + G_\theta(\beta) = G_\theta(\beta) + G_\theta(\alpha).$$

The veto map depends only on the resulting vector, so it preserves the equality. ■

Thus the million ordered interactions contain only

$$\frac{1000(1000+1)}{2} = 500{,}500$$

unordered interactions.

The ordered representation is nevertheless mathematically useful because it retains the direct-product indexing

$$A \times A = \mathbb{Z}_{10}^6.$$

**Definition 8. Coordinate permutation action**

For $\sigma \in S_3$, define

$$\sigma \cdot (\alpha_1, \alpha_2, \alpha_3) = \left(\alpha_{\sigma^{-1}(1)}, \alpha_{\sigma^{-1}(2)}, \alpha_{\sigma^{-1}(3)}\right). \qquad (44)$$

The same permutation acts on RGB vectors by permuting their coordinates.

**Proposition 8. Coordinate equivariance**

For every $\sigma \in S_3$,

$$G_\theta(\sigma \cdot \alpha) = \sigma \cdot G_\theta(\alpha), \qquad (45)$$

and

$$I_\theta(\sigma \cdot \alpha, \sigma \cdot \beta) = \sigma \cdot I_\theta(\alpha, \beta). \qquad (46)$$

**Proof**

The tuple lift applies the same scalar source $u_j(\theta)$ independently in each coordinate. Therefore permuting the indices merely permutes the resulting vector components. Since the cube $[0,1]^3$ is invariant under coordinate permutation, the veto commutes with the $S_3$-action. ■

The image therefore contains no mathematically privileged color coordinate. Red, green, and blue are coordinate labels imposed only at the rendering stage.

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**Definition 9. Diagonal cyclic translation**

For $q \in \mathbb{Z}_{10}$, define

$$T_q(\alpha_1, \alpha_2, \alpha_3) = (\alpha_1 + q, \alpha_2 + q, \alpha_3 + q)(\mathrm{mod} 10). \qquad (47)$$

**Theorem 5. Phase-index covariance**

For every $q \in \mathbb{Z}_{10}$,

$$G_{\theta + 2\pi q/10}(\alpha) = G_\theta(T_q \alpha), \qquad (48)$$

and

$$I_{\theta + 2\pi q/10}(\alpha, \beta) = I_\theta(T_q \alpha, T_q \beta). \qquad (49)$$

**Proof**

By Proposition 1,

$$u_j\left(\theta + \frac{2\pi q}{10}\right) = u_{j+q}(\theta).$$

Applying this identity independently to the three coordinates proves equation (48). Addition and veto then give equation (49). ■

Thus a phase advance of

$$\frac{2\pi}{10} = \frac{\pi}{5}$$

does not create a new abstract field. It cyclically relabels the existing field.

At the one-channel level, let $P$be the $10 \times 10$ permutation matrix implementing $j \mapsto j + 1$. Then

$$M_{\theta+\pi/5} = P^{-1} M_\theta P. \qquad (50)$$

For the complete support matrix,

$$\mathcal{M}_\theta = M_\theta^{\otimes 3},$$

one obtains

$$\mathcal{M}_{\theta+\pi/5} = (P^{-1})^{\otimes 3} \mathcal{M}_\theta P^{\otimes 3}. \qquad (51)$$

Hence phase evolution induces permutation similarity.

**Corollary 3. Phase invariance of algebraic observables**

The following quantities are invariant under

$$\theta \mapsto \theta + \frac{\pi}{5}:$$
$$\mathrm{rank}(M_\theta), \mathrm{tr}(M_\theta), \mathrm{spec}(M_\theta), m_\theta. \qquad (52)$$

The corresponding quantities for $M_\theta^{\otimes 3}$are also invariant.

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## 10. Tensor Rank and Spectrum

The factorization

$$\mathcal{M}_\theta = M_\theta^{\otimes 3} \qquad (53)$$

determines more than the admissible count. It transfers the complete linear algebra of the $10 \times 10$mask to the $1000 \times 1000$support matrix.

**Theorem 6. Rank multiplication**

Over any field for which the entries are interpreted algebraically,

$$\operatorname{rank}\left(M_\theta^{\otimes 3}\right) = \operatorname{rank}(M_\theta)^3. \qquad (54)$$

**Proof**

For finite matrices $A$and $B$,

$$\operatorname{rank}(A \otimes B) = \operatorname{rank}(A)\operatorname{rank}(B).$$

Applying this identity twice gives

$$\operatorname{rank}(M_\theta \otimes M_\theta \otimes M_\theta) = \operatorname{rank}(M_\theta)^3.$$

■

Thus a potentially large $1000 \times 1000$rank computation reduces exactly to a $10 \times 10$computation.

**Theorem 7. Spectral multiplication**

Let

$$\lambda_1, \dots, \lambda_{10}$$

be the eigenvalues of $M_\theta$, counted with algebraic multiplicity. Then the eigenvalues of $\mathcal{M}_\theta$are precisely

$$\lambda_i \lambda_j \lambda_k, 1 \le i, j, k \le 10. \qquad (55)$$

**Proof**

If

$$M_\theta v_i = \lambda_i v_i,$$

then

$$\begin{aligned} \mathcal{M}_\theta(v_i \otimes v_j \otimes v_k) &= (M_\theta v_i) \otimes (M_\theta v_j) \otimes (M_\theta v_k) \\ &= \lambda_i \lambda_j \lambda_k (v_i \otimes v_j \otimes v_k). \end{aligned}$$

The tensor eigenvectors span the full product space after passage to a splitting field or generalized eigenspaces. ■

**Corollary 4. Spectral radius**

$$\rho(\mathcal{M}_\theta) = \rho(M_\theta)^3. \qquad (56)$$

**Corollary 5. Trace law**

$$\mathrm{tr}(\mathcal{M}_\theta) = \mathrm{tr}(M_\theta)^3. \qquad (57)$$

**Corollary 6. Frobenius norm**

Because $M_\theta$is binary,

$$\| M_\theta \|_F^2 = m_\theta.$$

Therefore

$$\| \mathcal{M}_\theta \|_F^2 = m_\theta^3 = | \Sigma_\theta |. \qquad (58)$$

The admissible interaction count is consequently both a combinatorial cardinality and the squared Frobenius norm of the complete support tensor.

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**Definition 10. Scalar interaction matrix**

Let

$$S_\theta(i,j) = u_i(\theta) + u_j(\theta). \qquad (59)$$

Then

$$S_\theta = U_\theta \mathbf{1}^\top + \mathbf{1} U_\theta^\top, \qquad (60)$$

where $\mathbf{1} \in \mathbb{R}^{10}$ is the all-ones vector.

**Proposition 9. Low rank of the unrestricted scalar field**

$$\operatorname{rank}(S_\theta) \le 2. \qquad (61)$$

**Proof**

Equation (60) expresses $S_\theta$ as the sum of two rank-one matrices. ■

For nonconstant $U_\theta$,

$$\operatorname{rank}(S_\theta) = 2.$$

Thus the rich threshold geometry of $M_\theta$ arises by applying a nonlinear sign test to a rank-two matrix:

$$M_\theta = \mathbf{1}_{[0,\infty)}(S_\theta). \qquad (62)$$

This gives a compact structural summary:

$$\boxed{\text{rank-two scalar field} \overset{\text{threshold}}{\to} \text{binary mask} \overset{\otimes 3}{\to} \text{million-interaction support}} \qquad (63)$$

The apparent complexity is therefore produced by two operations: nonlinear thresholding and tensor exponentiation.

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### 11. Exact Reconstruction from One Hundred Numbers

The complete rendered field need not be stored as one million independently computed RGB values.

For each $(i,j) \in \mathbb{Z}_{10}^2$, retain only

$$S_\theta(i,j) = u_i(\theta) + u_j(\theta)$$

and

$$M_\theta(i,j) = \mathbf{1}_{[0,1]}(S_\theta(i,j)).$$

These are two $10 \times 10$tables.

For

$$\alpha = (i_1, i_2, i_3), \beta = (j_1, j_2, j_3),$$

define

$$\mu_\theta(\alpha,\beta) = \prod_{r=1}^{3} M_\theta\,(i_r, j_r). \qquad (64)$$

Then the complete field is reconstructed by

$$I_\theta(\alpha,\beta) = \mu_\theta(\alpha,\beta)(S_\theta(i_1,j_1), S_\theta(i_2,j_2), S_\theta(i_3,j_3)). \qquad (65)$$

**Theorem 8. Finite reconstruction theorem**

The million-cell RGB field $I_\theta$is determined exactly by the one-dimensional vector

$$U_\theta \in \mathbb{R}^{10}. \qquad (66)$$

Equivalently, it is determined by the 100-entry scalar interaction matrix $S_\theta$.

**Proof**

Equation (60) constructs $S_\theta$from $U_\theta$. Equation (62) constructs $M_\theta$from $S_\theta$. Equation (65) then reconstructs every RGB interaction. ∎

The logical state of the system therefore has constant finite size relative to its rendered output:

$$10 \longrightarrow 100 \longrightarrow 1000 \longrightarrow 1{,}000{,}000. \qquad (67)$$

The successive numbers count, respectively,

scalar samples,scalar interactions,RGB tuples,RGB interactions.

No randomness is required at any stage.

**Corollary 7. Output-sensitive complexity**

The mathematical field can be specified in

$$O(10^2)$$

scalar data, although explicit rasterization of all interactions requires

$$O(10^6)$$

output operations.

This distinction separates **generative complexity** from **display complexity**.

The generative law is small. The visible field is large.

## 12. Generalization, Dynamics, and Interpretation

The historical construction is one member of a larger deterministic family. Let

$$u_j(\theta) = b + a\sin\left(\theta + \frac{2\pi j}{n}\right) + h\sin\left(2\theta + \frac{4\pi j}{n}\right), j \in \mathbb{Z}_n, \qquad (68)$$

where

$$n \geq 2, a, b, h \in \mathbb{R}.$$

The associated tuple space is

$$A_n = \mathbb{Z}_n^d,$$

for arbitrary color or coordinate dimension $d \geq 1$.

For

$$\alpha, \beta \in A_n,$$

define

$$G_\theta(\alpha) = (u_{\alpha_1}(\theta), \dots, u_{\alpha_d}(\theta)) \qquad (69)$$

and

$$F_\theta(\alpha, \beta) = G_\theta(\alpha) + G_\theta(\beta). \qquad (70)$$

Let

$$C_d = [0,1]^d$$

and define the hard-veto field

$$I_\theta(\alpha, \beta) = \begin{cases} F_\theta(\alpha, \beta), & F_\theta(\alpha, \beta) \in C_d, \\ 0, & F_\theta(\alpha, \beta) \notin C_d. \end{cases} \qquad (71)$$

**Definition 11. General scalar mask**

Define

$$M_\theta^{(n)}(i, j) = \mathbf{1}_{[0,1]}(u_i(\theta) + u_j(\theta)), \qquad (72)$$

and let

$$m_\theta^{(n)} = \sum_{i,j \in \mathbb{Z}_n} M_\theta^{(n)}(i, j). \qquad (73)$$

**Theorem 9. $d$-dimensional tensor law**

The admissibility tensor in dimension $d$ is

$$\mathcal{M}_\theta^{(n \cdot d)} = \left(M_\theta^{(n)}\right)^{\otimes d}. \qquad (74)$$

Its support cardinality is

$$|\Sigma_\theta^{(n,d)}| = \left(m_\theta^{(n)}\right)^d. \qquad (75)$$

**Proof**

The cube condition is coordinatewise:

$$F_\theta(\alpha,\beta) \in [0,1]^d$$

if and only if

$$u_{\alpha_r}(\theta) + u_{\beta_r}(\theta) \in [0,1]$$

for every

$$r = 1, \dots, d.$$

The indicator of the conjunction is the product of the $d$scalar indicators. Hence the support tensor is the $d$-fold Kronecker product, and its number of unit entries is the $d$th power of the scalar count. ■

This gives the general expansion law

$$n \longrightarrow n^d \longrightarrow n^{2d}, \qquad (76)$$

where $n$scalar samples generate $n^d$tuples and $n^{2d}$ordered tuple interactions.

For the historical case,

$$(n,d) = (10,3),$$

so

$$10 \longrightarrow 10^3 \longrightarrow 10^6.$$

---

### 12.1 Phase evolution as a finite-state mask flow

Although $\theta$ varies continuously, the binary mask

$$M_\theta$$

changes only when at least one scalar sum reaches a boundary of the admissible interval.

For a general source, define the event set

$$\mathcal{B} = \{\theta : u_i(\theta) + u_j(\theta) \in \{0,1\} \text{ for some } i, j\}. \tag{77}$$

**Proposition 10. Piecewise constancy**

On every connected component of

$$\mathbb{R} \setminus \mathcal{B},$$

the mask $M_\theta$ is constant.

**Proof**

Each scalar sum

$$u_i(\theta) + u_j(\theta)$$

is continuous in $\theta$. Its membership in the open intervals

$$(-\infty, 0), (0,1), (1, \infty)$$

cannot change without crossing $0$or $1$. Therefore every mask entry is constant between boundary events. ■

The continuous animation therefore has two mathematically distinct layers:

continuous color motion

within a fixed support, and

discrete support transitions

at boundary phases.

For the historical source, the upper boundary is reached only at isolated maximal coincidences, whereas the principal support transitions occur at zero crossings.

The animation is consequently a finite-state symbolic flow driven by a continuous phase variable.

---

**Definition 12. Mask itinerary**

Let the event phases in one period be ordered as

$$0 \le \theta_1 < \theta_2 < \cdots < \theta_q < 2\pi.$$

Choose one phase

$$\phi_r \in (\theta_r, \theta_{r+1})$$

from each open interval, with cyclic indexing. The finite sequence

$$\mathcal{I} = \left(M_{\phi_1}, M_{\phi_2}, \ldots, M_{\phi_q}\right) \qquad (78)$$

is the **mask itinerary**.

The entire support dynamics is determined by $\mathcal{I}$ together with the boundary masks

$$M_{\theta_1}, \ldots, M_{\theta_q}.$$

For the historical generator, the generic admissible count remains $55$, but the locations of the admissible entries change as the phase crosses the event lattice.

Thus constant cardinality does not imply a static combinatorial field.

---

**12.2 Smooth veto deformation**

A hard veto introduces discontinuity at the cube boundary. The Observatory also admits a softness parameter that can be interpreted mathematically as replacing the indicator function with a continuous gate.

Let

$$q_\varepsilon : \mathbb{R} \longrightarrow [0,1]$$

be a boundary gate satisfying

$$\lim_{\varepsilon \to 0^+} q_\varepsilon(x) = \mathbf{1}_{[0,1]}(x)$$

at every $x \notin \{0,1\}$.

One possible choice is

$$q_\varepsilon(x) = \sigma\left(\frac{x}{\varepsilon}\right)\sigma\left(\frac{1-x}{\varepsilon}\right), \qquad (79)$$

where

$$\sigma(t) = \frac{1}{1+e^{-t}}.$$

Define the $d$-coordinate gate

$$Q_\varepsilon(x_1, \dots, x_d) = \prod_{r=1}^{d} q_\varepsilon\,(x_r). \qquad (80)$$

The soft-veto field is

$$I_{\theta,\varepsilon}^{\text{soft}}(\alpha,\beta) = Q_\varepsilon(F_\theta(\alpha,\beta))F_\theta(\alpha,\beta). \qquad (81)$$

**Proposition 11. Hard-veto limit**

For every interaction whose coordinates avoid the boundary values 0and 1,

$$\lim_{\varepsilon \to 0^+} I_{\theta,\varepsilon}^{\text{soft}}(\alpha,\beta) = I_\theta(\alpha,\beta). \qquad (82)$$

The softness control therefore does not define an unrelated visual effect. It supplies a regularized approximation to the discontinuous veto.

The hard and soft constructions occupy opposite ends of the family

$$\varepsilon > 0 \longrightarrow \varepsilon = 0.$$

The first is continuous and visually gradual. The second is exact and combinatorial.

---

**12.3 Harmonic perturbation**

The second-harmonic parameter $h$deforms the historical circle of scalar samples:

$$u_j(\theta) = b + a\sin\phi_j + h\sin 2\phi_j, \phi_j = \theta + \frac{2\pi j}{10}. \qquad (83)$$

The corresponding scalar interaction becomes

$$\begin{aligned} s_{ij}(\theta) = \quad & 2b + 2a\sin\left(\theta + \frac{\pi(i+j)}{10}\right)\cos\left(\frac{\pi(i-j)}{10}\right) \\ & +2h\sin\left(2\theta + \frac{\pi(i+j)}{5}\right)\cos\left(\frac{\pi(i-j)}{5}\right). \end{aligned} \qquad (84)$$

Equation (84) shows that the perturbation changes both the phase geometry and the dependence on index separation.

The historical antipodal law need not survive. Indeed,

$$u_{j+5}(\theta) = b - a\sin\phi_j + h\sin 2\phi_j,$$

whereas

$$-u_j(\theta) = -b - a\sin\phi_j - h\sin 2\phi_j.$$

Thus

$$u_{j+5} = -u_j$$

holds for every $j$and $\theta$only when

$$b = 0, h = 0. \qquad (85)$$

The historical setting is therefore a symmetry locus in the larger parameter space.

**Definition 13. Symmetry-breaking parameter plane**

Let

$$\mathcal{P} = \{(b,h) \in \mathbb{R}^2\}.$$

The antipodal symmetry locus is the single point

$$\mathcal{P}_{\text{anti}} = \{(0,0)\}. \qquad (86)$$

Motion away from $(0,0)$breaks exact sign pairing and permits scalar admissibility counts other than

$$\frac{100 + Z_\theta}{2}.$$

The Observatory controls therefore permit direct visual study of a symmetry-breaking deformation.

---

### 12.4 Amplitude and offset bounds

For the generalized source,

$$| u_j(\theta) - b | \leq | a | + | h |.$$

Hence

$$2b - 2(| a | + | h |) \leq u_i(\theta) + u_j(\theta) \leq 2b + 2(| a | + | h |). \qquad (87)$$

**Proposition 12. Universal admissibility regimes**

If

$$2b - 2(| a | + | h |) \geq 0$$

and

$$2b + 2(| a | + | h |) \leq 1,$$

then every scalar pair is admissible:

$$m_\theta = 100$$

for all $\theta$, and therefore

$$|\Sigma_\theta| = 10^6. \qquad (88)$$

If

$$2b + 2(|a| + |h|) < 0$$

or

$$2b - 2(|a| + |h|) > 1,$$

then no scalar pair is admissible:

$$m_\theta = 0$$

and

$$|\Sigma_\theta| = 0. \qquad (89)$$

These sufficient conditions identify parameter regions of total illumination and total veto.

Between these regions lies the nontrivial phase-dependent regime in which the admissibility topology evolves.

---

**12.5 Numerical exactness**

The historical antipodal cancellations satisfy

$$u_i(\theta) + u_{i+5}(\theta) = 0$$

as an algebraic identity.

Direct floating-point evaluation may instead produce values such as

$$-10^{-16}$$

or

$$10^{-16}.$$

A strict numerical test

$$s_{ij} \geq 0$$

can therefore misclassify theoretically zero interactions.

**Definition 14. Certified comparison**

A numerically evaluated scalar sum $\hat{s}_{ij}$is classified by

$$\mathrm{class}_\tau(\hat{s}_{ij}) = \begin{cases} -1, & \hat{s}_{ij} < -\tau, \\ 0, & |\hat{s}_{ij}| \leq \tau, \\ 1, & \hat{s}_{ij} > \tau, \end{cases} \tag{90}$$

where $\tau$is chosen larger than the verified floating-point error bound.

For the historical source, a stronger method is available: permanent antipodal pairs can be recognized symbolically from their indices,

$$j - i \equiv 5 (\mathrm{mod} 10),$$

and set to zero exactly.

The preferred audit hierarchy is therefore

$$\boxed{\text{symbolic identity} > \text{interval arithmetic} > \text{error-bounded tolerance} > \text{unqualified floating-point sign}}$$

The distinction matters because an error of only four scalar mask entries changes the three-dimensional count substantially. For example,

$$51^3 = 132{,}651,$$

whereas the exact generic count is

$$55^3 = 166{,}375.$$

The difference is

$$33{,}724$$

RGB interactions.

Small one-dimensional classification errors are amplified cubically.

---

**12.6 Mathematical interpretation**

Tuple Imagery does not merely assign colors to a list of numbers. It constructs a sequence of exact mathematical lifts:

$$\begin{aligned} &U_\theta \in \mathbb{R}^{10}, \\ &G_\theta \in (\mathbb{R}^3)^{\mathbb{Z}_{10}^3}, \\ &F_\theta \in (\mathbb{R}^3)^{\mathbb{Z}_{10}^6}, \end{aligned}$$

and

$$I_\theta \in ([0,1]^3)^{\mathbb{Z}_{10}^6}.$$

The construction may be summarized as

$$\mathbb{Z}_{10} \overset{u_\theta}{\to} \mathbb{R} \overset{(-)^{\times 3}}{\to} \mathbb{R}^3 \overset{+}{\to} \mathbb{R}^3 \overset{V}{\to} [0,1]^3. \qquad (92)$$

The apparent image is the visible projection of a six-index tensor field.

Its essential operations are:

cyclic sampling,
Cartesian lifting,
componentwise superposition,
convex-cube admissibility,

and

tensor-factorized rendering.

The million-interaction field is not an arbitrary accumulation of pixels. It is an exact finite object generated by repeated use of a single scalar law.

---

### 12.7 Conclusion

Tuple Imagery demonstrates how a low-dimensional periodic source can generate a high-cardinality visual field without losing mathematical auditability.

The historical ten-sample system begins with

$$u_j(\theta) = \frac{1}{2}\sin\left(\theta + \frac{2\pi j}{10}\right).$$

Its tuple lift generates

$$10^3 = 1000$$

ordered RGB sources, and their pairwise sums generate

$$10^6$$

ordered interactions.

Nevertheless, the complete admissibility structure is exactly

$$M_\theta^{\otimes 3},$$

where $M_\theta$ is only $10 \times 10$.

For generic phase,

$$m_\theta = 55,$$

so the exact number of admissible interactions is

$$55^3 = 166{,}375.$$

At singular phases, the count rises to either

$$59^3 = 205{,}379$$

or

$$60^3 = 216{,}000.$$

The mathematical essence of Tuple Imagery is therefore not the million-cell display itself. It is the compression law

$$\boxed{\text{one cyclic scalar source} \implies \text{one binary interaction mask} \implies \text{one tensor-powered color universe}}$$

or, in its shortest form,

$$\boxed{I_\theta = V\left[\left(U_\theta^{\times 3}\right)+\left(U_\theta^{\times 3}\right)\right], \operatorname{supp}_{\mathrm{adm}}(I_\theta) = M_\theta^{\otimes 3}.} \tag{94}$$

The image evolves continuously, its support changes discretely, and both are generated from a ten-component phase vector.

The supplementary video records this mathematical object as a dynamical visual field rather than as a static figure.

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